[12] These PROCEEDINGS, pp. 650–660 (Dec., 1931).

[13] Though it is very probable that the case C or C', in which our theorems hold is the general one for dynamical systems, it is not easy to construct effective examples. (See footnote 10.) An example, recently constructed by v. Neumann, will be published soon; it refers to a two-dimensional flow of the following type: the flow takes place in a rectangle, oriented parallel to the X and Y axes, the upper side of which has been replaced by suitable chosen curve $Y = F(X)$. The flow itself is parallel to the positive Y-axis, and each point X, $F(X)$ has to be identified with the corresponding point $X + \alpha$, 0. (The number $X + \alpha$ is to be taken mod. a, where a is the breadth of the parallelogram in the direction of the X-axis; α is a number incommensurable with a.) If $F(X)$ and α are suitably chosen this flow can be shown to fulfill C (and C').

PHYSICAL APPLICATIONS OF THE ERGODIC HYPOTHESIS

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I. In a recent issue of these PROCEEDINGS[1] the author has obtained a proof of the so-called quasi-ergodic hypothesis. The reader is referred to that paper for the precise formulation of this hypothesis, which plays so important a rôle in the foundations of classical statistical mechanics, and thus in the kinetic theory of gasses; the terminology of that paper will be used throughout this note. The exact statement of the mathematical result obtained in the previous paper by the author is as follows:

Let Ω be either the phase-space Φ of the mechanical system considered, or a sub space of Φ invariant under the transformation ($P \longrightarrow P_t$, P a point of Φ, t the time) induced by the equations of motion.[2] Let dv be the volume element defined in Ω invariant[3] under the transformation $P \longrightarrow P_t$, μN the Lebesgue measure (or weight) of $N(\subset \Omega)$ defined by means of dv : $\mu N = \int_N dv$. Let the time of sojourn of P_τ in N during the time $s < \tau < t$, divided by $t - s$, be denoted by $Z_{s,t}(N; P)$.

Then there exists a function $Z(N; P)$ such that, as $t - s \longrightarrow +\infty$, the function of P $Z_{s,t}(N; P)$ converges, in the sense of "strong convergence" in the space of functions of P, to the limit $Z(N; P)$; that is

$$\lim_{t-s\longrightarrow+\infty} \int_\Omega | Z_{s,t}(N; P) - Z(N; P) |^2 dv = 0. \tag{1}$$

This property determines $Z(N; P)$, which function, in our previous paper, is studied in more detail and calculated explicitly. The condition for the validity of the so-called quasi-ergotic hypothesis is that $Z(N; P)$ be independent of P; we have shown in our earlier paper that this will be true if and only if there exists in Ω no integral of the equations of motion other than one almost everywhere constant (for more details, cf. the paper in question).

This result was obtained on the basis of a method due to B. O. Koopman[4] in which the study of dynamical systems is undertaken with the aid of certain unitary and Hermitean functional operators, the spectral resolution of which is made use of systematically. Corresponding with the fact that this method operates with reference to function space, is the fact that our result (1) is in terms of "strong convergence," or convergence in the mean. A natural question to ask is whether our results, e.g., (1), could not be established in terms of convergence almost everywhere in Ω.

Mr. G. D. Birkhoff, to whom we communicated these results orally in October, 1931, has subsequently succeeded in establishing the above surmise by means of an extremely astute method of his own in the domain of point set theory; he has proved, i.e., the existence and equality of the numerical limits

$$\lim_{t \to +\infty} Z_{0,t}(N; P), \quad \lim_{s \to -\infty} Z_{s,0}(N; P)$$

except on a set of measure zero[5]—which limits, in virtue to (1), will be equal to $Z(N; P)$.

In view of these facts, it is of interest to decide which of the two formulations, (1) or (2), corresponds to the actual physical problem of the ergodic hypothesis. It turns out that the weaker form of statement (1) is sufficient, —that it, indeed, is the precise mathematical equivalent of the physical state of affairs. It is to be noted, further, that the knowledge of the spectral resolution $E(\lambda)$, which is fundamental in Koopman's method,[1,4] enables one to dominate the physical situation here completely; in particular, it furnishes a numerical estimation of the degree of convergence of the limiting process connected with the ergodic hypothesis, whereas Birkhoff's existence proof for (2) is of a non-constructive character.

II. The physical statement of the problem is as follows:

Consider a function $f(P)$ (in Ω) which is a physical quantity referring to the macroscopic state of the mechanical system (e.g., the pressure of a gas, the mean energy per degree of freedom or temperature, etc.). $f(P)$ changes with the time, having at the instant t the value $f(P_t)$, so that if the interval of time $s < \tau < t$ is so short that the time taken in the measurement fills it completely, the quantity measured is not $f(P)$ itself but its time average for $s < \tau < t$, viz., $\frac{1}{t - s}\int_s^t f(P_\tau)\, d\tau$. Is it, then, possible to find a constant C (constant with respect to P, C will naturally depend on f) by which $\frac{1}{t - s}\int_s^t f(P_\tau)\, d\tau$ may be replaced in every application without committing too great an error? (The fact that, if such a C exists at all, the "micro-canonical" mean $\frac{1}{\mu\Omega}\int_{\Omega} f(P)\,dv$ may be employed, is evident.)

The criterion for such a possibility is obviously the following: Can a constant C be so determined that the statistical dispersion of $\frac{1}{t-s}\int_s^t f(P_\tau)\,d\tau$ about C is small, i.e.,

$$\int_{\Omega}\left|\frac{1}{t-s}\int_s^t f(P_\tau)\,d\tau - C\right|^2 dv < \epsilon \qquad (3)$$

for a given $\epsilon > 0$? Now (1) states precisely that this is the case for $t - s$ sufficiently large, in the case where $f(P)$ is taken to be the characteristic function of the set $N \subset \Omega$:

$$f(P) = \chi_N(P)\begin{cases} = 1, \text{ for } P \text{ in } N \\ = 0, \text{ otherwise.}\end{cases}$$

From this the truth of (3) follows for all $f(P)$, (that is:

$$\lim_{t-s\longrightarrow+\infty}\int_{\Omega}\left|\frac{1}{t-s}\int_s^t f(P_\tau)\,d\tau - C\right|^2 dv = 0) \qquad (3')$$

(cf. note[1]); for the set of functions $f(P)$ for which it is valid forms a closed linear manifold in function space, and such a manifold contains every $f(P)$ if it contains every $\chi_N(P)$ of finite μN (cf. [1]); the integral of $|f(P)|$ must be finite, as is always the case in the applications.

The statement in terms of probability which corresponds to (2) may be made as follows (where the theorem of Egoroff has been used, in virtue of which any almost everywhere convergent sequence of functions will, for an arbitrary $\epsilon > 0$, converge uniformly except for a set of measure $\leqq \epsilon$[6]):

For every $\epsilon > 0$ and $\delta > 0$ there exists a $T = T(\epsilon, \delta)$ such that the occurrence of $\left|\frac{1}{t}\int_0^t f(P_\tau)\,d\tau - C\right| > \delta$ for any $t >$ is of probability $\leqq \epsilon$.

We have here a refinement of the statement that the statistical dispersion is small; but the latter is quite sufficient for all physical applications.

III. The fact that $t - s$ must be "short" in the physical application and "infinitely long" in the mathematical theorem ($t - s \longrightarrow + \infty$ was premised!) can be explained without inconsistency once the degree of convergence in (3′) is estimated. Such an estimation may be made with the aid of the method of Koopman.

In the notation of the earlier paper, (3′) states that

$$\lim_{t-s\longrightarrow+\infty} \|\sigma_{t,s} f - E_0 f\|^2 = 0$$

(cf. note,[1] for this and for the following), and the expression $|| \ldots ||^2$ on the left was calculated to be[7]

$$\int_{-\infty}^{-0} + \int_{+0}^{+\infty} \left(\frac{\sin \frac{1}{2}\lambda\,(t-s)}{\frac{1}{2}\lambda(t-s)}\right)^2 d\,||\,E(\lambda)f\,||^2.$$

It is a simple matter to evaluate this expression when $E(\lambda)$ is known. For example, if an interval $-\epsilon < \lambda < \epsilon$ exists, which contains no part of the continuous spectrum, and no point spectrum (besides the simple proper value 0), it is readily seen to be $= \dfrac{4\,||f\,||^2}{\epsilon^2(t-s)^2}$. In the general case various formulae can be obtained, for example,[8]

$$\left\|\left(E\left(\frac{1}{\sqrt{t-s}}\right) - E(+0)\right)f\right\|^2 + \left\|\left(E(-0) - E\left(-\frac{1}{\sqrt{t-s}}\right)\right)f\right\|^2 + \frac{4\,||f\,||^2}{t-s},$$

which furnishes an immediate evaluation. In the case mentioned first $\dfrac{||\,\sigma_{t,s}f - E_0 f\,||}{||\,f\,||}$ would be $\leqq \dfrac{2}{\epsilon(t-s)}$ this converging uniformly to 0 for all functions f.[9]

These evaluations could easily be analyzed further, but we shall leave the matter now. The example was only given to illustrate the use of Koopman's method in the setting of physical questions.

[1] *Proc. Nat. Acad. Sci.*, **18**, 1 (1932).

[2] That is, an "integral surface," e.g., an energy surface.

[3] In virtue of Liouville's theorem dv is the usual element of volume in the case $\Omega = \Phi$.

[4] *Proc. Nat. Acad. Sci.*, **17**, 5 (1931).

[5] *Ibid.*, **17**, 12 (1931).

[6] *Paris C. R.*, **152** (1911).

[7] The f appearing there is to be replaced by $f - E_0 f$ (E_0 is permutable with every $E(\lambda)$ and $\sigma_{t,s}$); this has for effect that $\int_{-\infty}^{-0} + \int_{+0}^{+\infty}$ replaces the earlier $\int_{-\infty}^{+\infty}$.

[8] Separate $\int_{-\infty}^{-0} + \int_{+0}^{+\infty}$ into $\left[\int_{+0}^{+\frac{1}{\sqrt{t-s}}} + \int_{-\frac{1}{\sqrt{t-s}}}^{-0}\right] + \left[\int_{+\frac{1}{\sqrt{t-s}}}^{+\infty} + \int_{-\infty}^{-\frac{1}{\sqrt{t-s}}}\right]$ and not that $\left(\frac{\sin \frac{1}{2}\lambda\,(t-s)}{\frac{1}{2}\lambda\,(t-s)}\right)^2 \leqq 1$ and $\leqq \left(\frac{2}{\lambda\,(t-s)}\right)^2$.

[9] It is easily shown, that this only occurs for periodical motions in which all paths have the same period.